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Green function on the q-symmetric space $SU_q(2)/U(1)$

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Abstract. Invariant distance on the non-commutative C^*-algebra $C(SU_q(2))$ is constructed and the generalized functions on the q-symmetric space $M = SU_q(2)/U(1)$ are introduced. The Green function and the kernel on M are derived. A path integration is formulated. The Green function for the free massive scalar field on the non-commutative Einstein space $R^1 \times M$ is presented.

1. Introduction

How the quantum mechanical effects should be altered if we replace the spacetime continuum with a non-commutative geometry is an exciting question. To answer this question we have to formulate the known quantum mechanical problems over non-commutative spaces. Since we lack satisfactory mathematical tools, construction of the Schrödinger equations over non-commutative spaces is difficult and sometimes arbitrary [1]. First, when we do not have a differentiable manifold it is problematic to find the correct operators replacing the derivatives. If, however, non-commutative geometry is given as a quantum group space this problem may be solved in a natural way since using the action of q-algebra generators one is not required to deal with q-differential calculus. If this is not the case, since it is always possible to build up an integration theory on a given set, the path integrals may, in principle, be a suitable method of quantization for non-commutative geometries in general. Therefore, the derivation of the Green functions over the non-commutative spaces is of interest. We should also remember that in usual quantum physics, defined over commutative spaces, the Green functions which are the vacuum (or temperature), expectation values of two-point field operators play an important role [2]. Even for q-group spaces the construction of the Green functions seems to be an important step in the formulation of many q-deformed quantum mechanical problems.

The experience we have in the derivation of Green functions over (undeformed) homogeneous spaces is quite rich; and it is also well known that many non-relativistic quantum mechanical problems are related to particle motion over these manifolds [3]. Therefore, we hope that by constructing the Green functions on the q-group spaces this may lead to meaningful definitions of these problems over non-commutative geometries. It is important to stress that if we know the formulations of the non-relativistic potential problems we also gain insight into some field theoretical effects. In fact, the calculations

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of many field theoretical problems, like the pair creations in given cosmologies or in external electromagnetic fields, and Casimir interactions, may formally become equivalent to some non-relativistic potential problems. For example, to investigate pair production in the Robertson–Walker spacetime expanding with the factor $a(t)$ one has to calculate the Green function of a particle moving in the one-dimensional potential $V(t) = a^{-2}(t)$ with the time t playing the role of coordinate [4].

Motivated by the considerations summarized above, construction of the Green functions over quantum homogeneous spaces is the subject of this work. The specific example we study is the quantum symmetric space $M = SU_q(2)/U(1)$. Quantum symmetric spaces have already been the subject of some interesting investigations. For example, the well known relations between special functions and classical groups have been generalized to quantum groups [5]. Studies related to the quantum spheres already exist in the literature [6]. Schrödinger equations in connection with quantum group symmetry have also attracted attention [7]. Recently the homogeneous space of $E_q(2)$ has been considered and the q-Schrödinger equation on it constructed [8].

In section 2, after a brief review of the invariant distance concept, we outline a method for constructing the Green functions by two classical group examples. In this method, which is applicable to the quantum groups, one first constructs the one-point 'Green function', then obtains the Green function depending on two points by the group action.

In section 3 the invariant distance for the quantum group $A = \mathrm{Pol}(SU_q(2))$ and the q-symmetric space $M = SU_q(2)/U(1)$ is constructed and its properties are demonstrated.

In the derivation of the Green function we have to construct the class of generalized functions on the q-symmetric space M. Section 4 is devoted to this construction.

In section 5 the one-point 'Green function' is derived on the space M, from which we obtain the Green function in section 6.

In section 7 we introduce the time development kernel on the space M. Having this kernel in hand the non-commutative path integration is also formulated.

Finally, in section 8 the Green function for the massive scalar field on $R^1 \times M$, which is the non-commutative version of the Einstein space, is constructed.

The basic definitions and the established results about the Hopf algebra A which we use in our work are given in the appendices.

2. Method for constructing Green functions. Examples from classical Lie groups

The Green function of the free particle motion over a Lie group manifold and its homogeneous spaces depend on the invariant distance between two points. When we attempt to construct Green functions over the quantum homogeneous spaces the first problem we have to face is the introduction of the invariant distance. To overcome this problem it is instructive to review the case of classical Lie groups for the purpose of developing a method for constructing Green functions which can also be employed for quantum groups. We briefly study two examples.

We first consider the real line $\mathcal{R}$ which is the homogeneous space with respect to the translation group $T(x)$. The Green function of the free particle motion over $\mathcal{R}$ depends on the invariant distance $|x - x'|$ and on the momentum $p \in (-\infty, \infty)$ which is the weight of the fundamental unitary irreducible representation. The homogeneity of $\mathcal{R}$ under the action of $T(x)$ implies

$$\mathcal{G}^p(x, x') = T^{-1}(x')\mathcal{G}^p(x, 0). \tag{2.1}$$

This equation suggests that the invariant Green function on the homogeneous space can be obtained by group action if the one-point 'Green function' $\mathcal{G}^p(x,0)$ is known.

As the second example we consider the classical symmetric space $SU(2)/U(1)$. If an element $g \in SU(2)$ is parametrized as $g = k(\psi)a(\theta)k(\psi')$ with

$$k(\psi) = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \qquad a(\theta) = \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{2.2}$$

the symmetric space M which is topologically equivalent to S^2 is represented as

$$x = g\sigma(g^{-1}) \qquad x \in M. \tag{2.3}$$

Here σ is the involutive automorphism having the property

$$\sigma(k) = k \qquad \sigma(a) = a^{-1}. \tag{2.4}$$

The Green function $\mathcal{G}$ on M depends on the group invariants which are the weights of the fundamental unitary irreducible representations $l = 0, 1, 2, \ldots,$ and on the invariant distance given by

$$\rho(x_1, x_2) = 1 - \tfrac{1}{2}\,\mathrm{Tr}(x_1 x_2^{-1}) \tag{2.5}$$

with $x_1, x_2 \in M$. Since the point of the symmetric space is given by the two-by-two matrix (2.3), the symbol x^{-1} means the inversion of the matrix x. Using the group property $g_2^{-1}g_1 = g_{12}$ we can write

$$\begin{aligned}\mathrm{Tr}(x_1x_2^{-1}) &= \mathrm{Tr}(g_1\sigma(g_1^{-1})(g_2\sigma(g_2^{-1}))^{-1}) = \mathrm{Tr}(g_2^{-1}g_1\sigma((g_2^{-1}g_1)^{-1})) \\ &= \mathrm{Tr}(g_{12}\sigma(g_{12}^{-1})) = \mathrm{Tr}(x_{12}).\end{aligned} \tag{2.6}$$

If we fix one of the points as $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain a formula depending on only one point. Taking advantage of this formula we can first construct a one-point 'Green function', then by the action of the group element we arrive at the Green function which is dependent on two points. The equation satisfied by the one-point 'Green function' is

$$(\hat{\mathcal{C}} - (l + \tfrac{1}{2})^2)\mathcal{G}^l(x) = \delta(x) \tag{2.7}$$

where $\hat{\mathcal{C}}$ is the Casimir element. Once we obtain the solution of this equation we can derive the Green function simply by the group action as

$$\mathcal{G}^l(x_1, x_2) = T(g_2^{-1})\mathcal{G}^l(x_1) = \mathcal{G}^l(g_2x_1\sigma(g_2^{-1})) \tag{2.8}$$

which satisfies

$$(\hat{\mathcal{C}} - (l + \tfrac{1}{2})^2)\mathcal{G}^l(x_1, x_2) = \delta(x_1 - x_2). \tag{2.9}$$

3. An invariant distance on the quantum group $SU_q(2)$

The coordinate functions a, a^*, b and b^* of the Hopf $*$-algebra A (see appendix A) satisfy the commutation relations [9]

$$b^*b = bb^* \qquad ba = qab \qquad b^*a = qab^* \qquad aa^* + b^*b = 1 \qquad a^*a + q^2bb^* = 1. \tag{3.1}$$

Any $*$-representation of the C^*-algebra $C(SU_q(2))$ which is the suitable completion of the algebra of polynomials on $SU_q(2)$ [10] is either one-dimensional or unitary equivalent to the following representation written for $q < 1$

$$\begin{aligned}&\pi_\phi(a)|n\rangle = (1-q^{2n})^{\frac{1}{2}}|n-1\rangle \qquad \pi_\phi(b)|n\rangle = e^{i\phi}q^n|n\rangle \qquad \pi_\phi(b^*)|n\rangle = e^{-i\phi}q^n|n\rangle \\ &\pi_\phi(a^*)|n\rangle = (1-q^{2n+2})^{\frac{1}{2}}|n+1\rangle\end{aligned} \tag{3.2}$$

which is irreducible for fixed $\phi \in (0, 2\pi]$ [11]. Since the representation (3.2) exhaust all the non-equivalent irreducible $*$-representations of the C^*-algebra $C(SU_q(2))$ we will omit π_ϕ in the formulae of the next sections.

In the $q \to 1$ limit the relations (3.1) define the three-dimensional space which is the manifold of $SU(2)$. To understand the situation in the deformed case we recall the usual quantum mechanics in which the physical systems are defined by vectors in the Hilbert space and the self-adjoint operators correspond to the observables including coordinates. In the same manner for quantum groups we construct self-adjoint operators X from linear combinations of coordinate functions and define the expectation values of these operators as the points of the quantum group space as

$$X_\psi = \langle\psi|X|\psi\rangle \qquad X \in A, \psi \in \mathcal{H} \tag{3.3}$$

where the Hilbert space $\mathcal{H}$ is the carrier space of the representation (3.2). This definition can be carried to co-product space $X \otimes Y$ with $X,\ Y \in A$ to define the invariant distance for A which should go to the corresponding classical limit as $q \to 1$ and should have the following properties

$$\rho(g_1, g_2) = \rho(g_2, g_1) \tag{3.4}$$

$$\rho(g_1 g, g_2 g) = \rho(g_2, g_1) \tag{3.5}$$

$$\rho(g_1, g_2) > 0 \tag{3.6}$$

$$\rho(g, g) = 0 \tag{3.7}$$

$$\rho(g_1, g_2) < \rho(g_1, g_3) + \rho(g_3, g_2). \tag{3.8}$$

Motivated by formula (2.5) of the previous section we suggest the following Hermitian operator in $\mathcal{H} \otimes \mathcal{H}$

$$R = (\tau \otimes S)\Delta\left(1 - \frac{1}{[2]_q}\mathrm{Tr}_q(d^{\frac{1}{2}})\right). \tag{3.9}$$

Here $d^{\frac{1}{2}}$ is the matrix of the unitary irreducible co-representation of the Hopf algebra A with weight $\frac{1}{2}$ (see appendix A). $[\cdot]_q$ is defined as $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and the q-trace is given by

$$\mathrm{Tr}_q(d^{\frac{1}{2}}) = \sum_{j=-\frac{1}{2}}^{\frac{1}{2}} q^{-2j}(d^{\frac{1}{2}}_{jj}). \tag{3.10}$$

τ is the automorphism of A defined as

$$\tau(d^{\frac{1}{2}}) = \begin{pmatrix} q^{-1}a & b \\ -qb^* & qa^* \end{pmatrix}. \tag{3.11}$$

Note that in the $q \to 1$ limit the invariant distance (3.9) reduces to the ordinary invariant distance (2.5). Expectation values of the operator R of (3.9) in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ defines the correct invariant distance on $SU_q(2)$. In fact, this operator possesses all the properties of (3.4–3.8).

(i) The symmetry condition of (3.4) is fulfilled for

$$\sigma(R) = R. \tag{3.12}$$

Here σ is the flip homomorphism $\sigma(x \otimes y) = y \otimes x;\ x, y \in A$.

(ii) The invariance condition of (3.5) takes the form of

$$\langle\Delta \otimes \Delta R\rangle_4 = R \tag{3.13}$$

where $\langle\cdot\rangle_4$ is the map from $A \otimes A \otimes A \otimes A$ into $A \otimes A$ and is given by

$$\langle a \otimes b \otimes c \otimes d\rangle_4 = (a \otimes c)\psi(b\tau^{-1}(d)) \tag{3.14}$$

with ψ being the invariant integral on the quantum group A (see appendix B).

(iii) The operator R is positive. To show this we have to construct the basis in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ in which it is diagonal. We choose the eigenfunctions as

$$\psi_l = \sum_{j=-l}^{l} v_j^l |l + j\rangle \otimes |l - j\rangle. \tag{3.15}$$

We then sandwich the operator R of (3.9) between these states to get the recurrence relations for the unknown coefficients v_j^l

$$[2]_q v_j^l \left(1 - \frac{2q^{2l+1}}{[2]_q} - E_l\right) = v_{j-1}^l [l + j][l - j + 1] + v_{j+1}^l [l - j][l + j + 1]. \tag{3.16}$$

These coefficients are normalized as

$$\sum_{j=-l}^{l} \overline{v_j^l} v_j^l = 1. \tag{3.17}$$

E_l in (3.16) is the spectrum of the self-adjoint operator R and $[x]$ is defined as $[x] = (1 - q^{2x})^{1/2}$. As an example, consider the eigenvalue and eigenfunction for the $l = 0$ state

$$\psi_0 = |0\rangle \otimes |0\rangle. \tag{3.18}$$

The corresponding eigenvalue

$$E_0 = \frac{q^{-1} - q}{q^{-1} + q} \tag{3.19}$$

is positive for $q < 1$. Similar demonstrations can be done for all other values of l to prove that E_l is positive.

(iv) Condition of (3.7) is also satisfied for

$$m(\tau^{-1} \otimes \mathrm{id})R = 0 \tag{3.20}$$

where m is the operation of multiplication in the C^*-algebra $C(SU_q(2))$ and τ^{-1} is the inverse of the involution (3.11).

(v) Finally, the triangular inequality of (3.8) reads

$$\langle\Psi|(\mathrm{id} \otimes \sigma)(R \otimes 1)|\Psi\rangle < \langle\Psi|(R \otimes 1 + 1 \otimes R)|\Psi\rangle \qquad |\Psi\rangle \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}. \tag{3.21}$$

As in case (iii) we can show that the self-adjoint operator Π which is defined as

$$\Pi = R \otimes 1 + 1 \otimes R - (\mathrm{id} \otimes \sigma)(R \otimes 1) \tag{3.22}$$

is positive. For example, for the eigenfunction

$$\Psi_0 = |0\rangle \otimes |0\rangle \otimes |0\rangle \qquad \langle\Psi_0 \mid \Psi_0\rangle = 1 \tag{3.23}$$

the corresponding eigenvalue which is given by

$$E_0 = \frac{q^{-1} - q}{q^{-1} + q} \tag{3.24}$$

is positive for $q < 1$.

Before closing this section we give the invariant distance on the co-set space $M = A/K$ where $K = \mathrm{Pol}(U(1))$ (see appendix B). First, we introduce the involutive automorphism

$$\beta(d^{\frac{1}{2}}) = \begin{pmatrix} a & -b \\ qb^* & a^* \end{pmatrix}. \tag{3.25}$$

It is clear that it does not change the quantum subgroup K (see appendix A). By virtue of this automorphism we introduce the coordinate functions on the q-symmetric space $M = A/K$ as

$$t_{ij} = (d^{\frac{1}{2}}\beta(S(d^{\frac{1}{2}})))_{ij}. \tag{3.26}$$

The operator of the invariant distance on M is then given by

$$R_M = (\tau \otimes S)\Delta\left(1 - \frac{1}{[2]_q}\mathrm{Tr}_q(t)\right). \tag{3.27}$$

Introducing (3.26) into this formula we obtain

$$R_M = (\tau \otimes S)\Delta\xi \tag{3.28}$$

where $\xi = bb^* \in A$, which generates the two-sided co-set space $H = K\backslash A/K$ (see appendix A). In a fashion parallel to R one can check that R_M satisfies all the properties of the invariant distance.

4. Generalized functions on the q-symmetric space M

The Hopf $*$-algebra A is the quantized algebra of polynomials on $SU_q(2)$. Since our aim is the introduction of the Green function on the q-symmetric space $M = SU_q(2)/U(1)$ this class of polynomials is not enough. We have to define generalized functions on these non-commutative spaces. In this section we construct the class of generalized functions on the quantum sphere M.

Matrix elements of the unitary co-representation of A with weight $l = 1$ are the coordinate functions which generate the algebra of functions over the quantum sphere M. They are given by (see appendix B)

$$d^1_{0,1} = -qb^*a^* \equiv z \qquad d^1_{0,-1} = ab \equiv q^{-1}z^* \qquad d^1_{0,0} = 1 - (q^{-1} + q)\xi. \tag{4.1}$$

The commutation relations satisfied are

$$z\xi = q^2\xi z \qquad z^*\xi = q^{-2}\xi z^* \qquad zz^* - q^4 z^* z = (q^2 - 1)\xi. \tag{4.2}$$

Using the $*$-representation of the C^*-algebra $C(SU_q(2))$ given by (3.2) we get

$$\begin{aligned} z^*|n\rangle &= \mathrm{e}^{\mathrm{i}\psi} q^n (1 - q^{2n})^{\frac{1}{2}} |n-1\rangle \qquad \xi|n\rangle = q^{2n}|n\rangle \\ z|n\rangle &= \mathrm{e}^{-\mathrm{i}\psi} q^{n+1} (1 - q^{2n+2})^{\frac{1}{2}} |n+1\rangle. \end{aligned} \tag{4.3}$$

Any element $p \in M$ can uniquely be represented as the finite sum as

$$p = \sum_{n>0}^{N} z^n p_n(\xi) + \sum_{n>0}^{M} p_{-n}(\xi) z^{*n} + p_0(\xi) \tag{4.4}$$

with $p_n(\xi)$ being the polynomials in ξ. If in place of these polynomials we put functions $f_n(\xi)$ with finite support such that supp $(f_n) \subset \{q^2, q^4, q^6, \ldots\} = q^{2Z_+}$ we arrive at a vector space $\mathcal{F}(M)$. One can supply the vector space $\mathcal{F}(M)$ with a topology in which the matrix elements of the representation $\{\langle n|p|m\rangle\}_{n,m\in Z_+}$ are continuous. It is then possible to arrive at the space of generalized functions $\hat{\mathcal{F}}(M)$ which is the completion of $\mathcal{F}(M)$.

The space $\hat{\mathcal{F}}(M)$ contains the functions which can be represented as

$$f = \sum_{n>0} z^n f_n(\xi) + \sum_{n>0} f_{-n}(\xi) z^{*n} + f_0(\xi) \tag{4.5}$$

with supp $(f_n) \subset q^{2Z_+}$.

Repeating this procedure for the algebra $M \otimes M$ we obtain the space of generalized functions $\hat{\mathcal{F}}(M \otimes M)$ on the tensor product of quantum spheres.

5. One-point 'Green function'

We will follow the method we introduced in section 2 for the construction of the Green function over M. First, we have to obtain the one-point 'Green function' $\mathcal{G}_q^l(\xi)$. It is defined by the deformation of (2.7) as

$$(\hat{\mathcal{C}} - [l + \tfrac{1}{2}]_q^2)\mathcal{G}_q^l(\xi) = \delta_q(\xi) \tag{5.1}$$

where $\hat{\mathcal{C}}$ is the Casimir element (see appendix C). The invariant q-delta function $\delta_q(\xi)$ is a linear functional over the two-sided co-set space H which for any function $f \in A[0,0]$ satisfies

$$\langle \delta_q(\xi) \mid f(\xi) \rangle = f(0) \qquad f \in A[0,0] \tag{5.2}$$

where the scalar product is the one given in appendix B. It is easy to verify that the q-delta function can be represented as

$$\delta_q(\xi) = \sum_{l=0}^{\infty} [l + \tfrac{1}{2}]_q d_{0,0}^l(\xi) \tag{5.3}$$

where $d_{00}^l(\xi)$ is the q-zonal function (see appendix B)

$$d_{00}^l(\xi) = {}_2\phi_1(q^{-2l}, q^{2(l+1)}, q^2 \mid q^2, q^2\xi). \tag{5.4}$$

We can also verify by direct substitution that the one-point 'Green function' of (5.1) can be represented as

$$\mathcal{G}_q^l(\xi) = \sum_{n=0}^{\infty} \left[n + \frac{1}{2}\right]_q \frac{d_{0,0}^n(\xi)}{[n + \frac{1}{2}]_q^2 - [l + \frac{1}{2}]_q^2}. \tag{5.5}$$

The above summation can be executed to give an expression in terms of the q-hypergeometric function

$$\mathcal{G}_q^l(\xi) = \gamma^l \xi^{-l-1}\, {}_2\phi_1(q^{-2(l+1)}, q^{-2(l+1)}, q^{-4(l+1)} \mid q^{-2}, q^{-2}\xi^{-1}) \tag{5.6}$$

where γ^l is the normalization constant

$$\gamma^l = q^{-2l-1} \frac{\Gamma_{q^{-2}}(l+1)^2}{\Gamma_{q^{-2}}(2l+2)} \tag{5.7}$$

and where Γ is the q-gamma function

$$\Gamma_q(\nu) = (1-q)^{1-\nu} \frac{(q;q)_\infty}{(q^\nu, q)_\infty}. \tag{5.8}$$

The inverse of the element ξ can be represented as an infinite series which is the generalized function given by

$$\xi^{-1} = \sum_{n=0}^{\infty} (1-\xi)^n. \tag{5.9}$$

In the $q \to 1$ limit the Green function (5.6) becomes a Legendre function Q^l of the second kind [12].

6. Green function over $M = A/K$

Having the one-point Green function in hand we can now introduce the Green function $\mathcal{G}_q^l(x \otimes x)$ on the q-symmetric space $x \in M = A/K$ as

$$\mathcal{G}_q^l(x \otimes x) = (\tau \otimes S)\Delta \mathcal{G}_q^l(\xi) \tag{6.1}$$

where $\mathcal{G}_q^l(x \otimes x) \in \hat{\mathcal{F}}(M \otimes M)$. The equation satisfied by this Green function is

$$(\mathrm{id} \otimes \{\hat{\mathcal{C}} - [l + \tfrac{1}{2}]_q^2\})\mathcal{G}_q^l(x \otimes x) = \delta_q(x \otimes x) \tag{6.2}$$

where the invariant q-delta function is given by

$$\delta_q(x \otimes x) = (\tau \otimes S)\Delta \delta_q(\xi). \tag{6.3}$$

Note that $\mathcal{G}_q^l(\xi)$ is actually in the subspace $\widehat{\mathcal{F}(H)} \subset \widehat{\mathcal{F}(M)}$. Thus, we can apply co-multiplication on $\mathcal{G}_q^l(\xi)$ without any problem [13].

Substituting (5.3) into (6.3) and (5.5) into (6.1) we obtain the following representations for the invariant q-delta function and the Green function

$$\delta_q(x \otimes x) = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} \left[l + \frac{1}{2}\right]_q \tau(d_{0,j}^l(x)) \otimes \overline{d_{0,j}^l}(x) \tag{6.4}$$

$$\mathcal{G}_q^l(x \otimes x) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left[n + \frac{1}{2}\right]_q \frac{\tau(d_{0,j}^n(x)) \otimes \overline{d_{0,j}^n}(x)}{\left[l + \frac{1}{2}\right]_q^2 - \left[n + \frac{1}{2}\right]_q^2}. \tag{6.5}$$

Using the representation (5.6) of the one-point 'Green function' we have another expression for $\mathcal{G}_q^l(x \otimes x)$ in terms of the q-hypergeometric function as

$$\mathcal{G}_q^l(x \otimes x) = \gamma^l(\tau \otimes S)\Delta(\xi^{-l-1}\ {}_2\phi_1(q^{-2(l+1)}, q^{-2(l+1)}, q^{-4(l+1)} \mid q^{-2}, q^{-2}\xi^{-1})). \tag{6.6}$$

For any operator function $f(x)$, $x \in M$ and the linear operator P of the dual Hopf $*$-algebra $U(su_q(2))$ (see appendix C) the q-delta function of (6.4) satisfies

$$\langle \delta_q(x \otimes x) \mid \mathrm{id} \otimes f(x)\rangle_2 = f(x) \tag{6.7}$$

and

$$\langle (\hat{P} \otimes \mathrm{id})\delta_q(x \otimes x) \mid \mathrm{id} \otimes f(x)\rangle_2 = \langle \delta_q(x \otimes x) \mid \mathrm{id} \otimes \hat{P}^* f(x)\rangle_2 \tag{6.8}$$

where $\hat{P}$ is representative of P in M and $\langle\cdot|\cdot\rangle_2$ is the inner product defined as

$$\langle x_1 \otimes x_2 \mid y_1 \otimes y_2\rangle_2 = x_1 y_1 \psi(x_2 \tau^{-1}(y_2)) \qquad x_1, x_2, y_1, y_2 \in M \tag{6.9}$$

which is a map

$$(M \otimes M) \times (M \otimes M) \to M. \tag{6.10}$$

Before closing this section we would like to consider the inhomogeneous equation for a given constant E and operator function $f(x) \in M$

$$(\hat{\mathcal{C}} - E)\mathcal{F}(x) = f(x). \tag{6.11}$$

As in the classical case the solution is obtained by using the Green function $\mathcal{G}_q^E$

$$\mathcal{F}(x) = F_0(x) + \langle \mathcal{G}_q^E(x \otimes x) \mid f(x) \otimes \mathrm{id}\rangle_2 \tag{6.12}$$

where $F_0(x)$ is the complete solution of the homogeneous equation. It is obvious from the notation that $\mathcal{G}_q^E$ is the solution of the same equation as (6.2) with $[l + \frac{1}{2})]^2$ replaced by E.

7. Kernel on the q-symmetric space M

We introduce the unitary operator in terms of the real time interval t and the centre of the enveloping algebra $\hat{\mathcal{C}}$ as

$$U(t) = \mathrm{e}^{\mathrm{i}t\hat{\mathcal{C}}} \tag{7.1}$$

satisfying the semigroup property

$$U(t)U(t') = U(t+t'). \tag{7.2}$$

The one-point 'q-kernel' is then given by

$$\mathcal{K}_q(\xi, t) = U(t)\delta_q(\xi). \tag{7.3}$$

Inserting the representation of the one-point q-delta function from (5.3) we obtain

$$\mathcal{K}_q(\xi, t) = \sum_{l=0}^{\infty} \left[l + \frac{1}{2}\right]_q \mathrm{e}^{\mathrm{i}t\left[l+\frac{1}{2}\right]_q^2} d_{00}^l(\xi) \tag{7.4}$$

which is connected to the one-point 'Green function' through the relation

$$\mathcal{K}_q(\xi, t) = \int_{-\infty}^{\infty} \mathrm{d}E\, \mathrm{e}^{\mathrm{i}tE} \mathcal{G}_q^E(\xi). \tag{7.5}$$

It is obvious that the above kernel satisfies the equation

$$(\mathrm{i}\partial_t + \hat{\mathcal{C}})\mathcal{K}_q(\xi, t) = 0. \tag{7.6}$$

The two-point q-kernel is defined in a manner parallel to the definition of the one-point 'q-kernel' as

$$\mathcal{K}_q(x \otimes x, t) = (U(t) \otimes \mathrm{id})\delta_q(x \otimes x). \tag{7.7}$$

Inserting the representation of the two-point q-delta function of (6.4) into the above equation we have

$$\mathcal{K}_q(x \otimes x, t) = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} \mathrm{e}^{\mathrm{i}t\left[l+\frac{1}{2}\right]_q^2} \left[l + \frac{1}{2}\right]_q \tau(d_{0,j}^l(x)) \otimes \overline{d_{0,j}^l(x)}. \tag{7.8}$$

The triple invariant product $\langle \cdot \mid \cdot \rangle_3$ is defined by

$$\langle x_1 \otimes x_2 \otimes x_3 \mid y_1 \otimes y_2 \otimes y_3 \rangle_3 = x_1 y_1 \otimes x_3 y_3 \phi(x_2 \tau^{-1}(y_2)) \tag{7.9}$$

which is the map

$$(M \otimes M \otimes M) \times (M \otimes M \otimes M) \to M \otimes M \tag{7.10}$$

and enables us to derive the important property of $K_q(x \otimes x, t)$

$$\langle \mathcal{K}_q(x \otimes x, t) \otimes 1 \mid 1 \otimes \mathcal{K}_q(x \otimes x, t') \rangle_3 = \mathcal{K}_q(x \otimes x, t + t'). \tag{7.11}$$

Using this property we can introduce the path integral representation for the kernel on non-commutative space. Indeed from (7.11) it follows

$$\mathcal{K}_q(x \otimes x, T) = \langle \mathcal{K}_q(x \otimes x, T/2) \otimes 1 \mid 1 \otimes \mathcal{K}_q(x \otimes x, T/2) \rangle_3 \tag{7.12}$$

and

$$\mathcal{K}_q(x \otimes x, T/2) = \langle \mathcal{K}_q(x \otimes x, T/4) \otimes 1 \mid 1 \otimes \mathcal{K}_q(x \otimes x, T/4) \rangle_3. \tag{7.13}$$

Inserting (7.13) into (7.12) and using the shorthand notation $\mathcal{K}_q(T) = \mathcal{K}_q(x \otimes x, T)$ we get

$$\mathcal{K}_q(T) = \langle \langle \mathcal{K}_q(T/4) \otimes 1 \mid 1 \otimes \mathcal{K}_q(T/4) \rangle_3 \otimes 1 \mid \langle \mathcal{K}_q(T/4) \otimes 1 \mid 1 \otimes \mathcal{K}_q(T/4) \rangle_3 \otimes 1 \rangle_3. \tag{7.14}$$

Continuing the above process n-times and taking the limit $n \to \infty$ we arrive at a path integral formula

$$\mathcal{K}_q(x \otimes x, T) = \lim_{n\to\infty} \{(\mathcal{K}_q(x \otimes x, T/2^n)\}_n \tag{7.15}$$

where $\{\cdot\}_n$ stands for n-times repeated $\langle\cdot\rangle_3$ map.

8. Green function for the massive free scalar field on the q-Einstein space $R^1 \times M$

For the commutative translation group parametrized by t, the commutative and co-commutative Hopf algebra of $\mathrm{Fun}(t)$ is given by

$$\delta t = 1 \otimes t + t \otimes 1 \qquad S(t) = -t \qquad \varepsilon(t) = 0. \tag{8.1}$$

The one-point kernel for the free particle motion on (t, s) 'spacetime' is the usual one

$$\mathcal{K}(t, s) = (-4\mathrm{i}\pi s)^{-\frac{1}{2}} \mathrm{e}^{-t^2/4s}. \tag{8.2}$$

The kernel on $(R^1 \times M, s)$ is expressed as

$$\mathcal{K}(\xi, t, s) = \mathcal{K}(t, s)\mathcal{K}_q(\xi, s) \tag{8.3}$$

where $\mathcal{K}_q(\xi, s)$ is given by (7.4). Using the Schwinger–DeWitt representation

$$\mathcal{G}(\xi, t; m^2) = -\mathrm{i}\theta(t) \int_0^\infty \mathrm{e}^{-\mathrm{i}m^2 s} \mathcal{K}(\xi, t, s)\, \mathrm{d}s \tag{8.4}$$

with $\mathrm{Im}(m^2) < 0$ and $\theta(t)$ being the step function we get the one-point 'Green function' over $R^1 \times M$ for the scalar field with mass m. Performing integration over $\mathrm{d}s$ we obtain

$$\mathcal{G}(\xi, t; m^2) = \theta(t) \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}t\sqrt{[n+\frac{1}{2}]_q + m^2}} \frac{[n + \frac{1}{2}]_q}{\sqrt{[n + \frac{1}{2}]_q^2 + m^2}} d_{00}^n(\xi). \tag{8.5}$$

The above Green function satisfies

$$(\partial_t^2 - \hat{\mathcal{C}} + m^2)\mathcal{G}(\xi, t; m^2) = \delta(t)\delta_q(\xi). \tag{8.6}$$

Following the procedure of section 5 we obtain the invariant Green function on the space $R^1 \times M$ depending on two points

$$\mathcal{G}(y \otimes y, ; m^2) = (\tau \otimes S)\Delta\mathcal{G}(\xi, t; m^2) \tag{8.7}$$

where $y \in R^1 \times M$ and the operations Δ and τ on t are given by $\Delta t = \delta t$ and $\tau(t) = t$. The Green function (8.7) satisfies

$$(\mathrm{id} \otimes (\partial_t^2 - \hat{\mathcal{C}} + m^2))\mathcal{G}(y \otimes y; m^2) = \delta(t \otimes 1 - 1 \otimes t)\delta_q(x \otimes x). \tag{8.8}$$

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Appendix A. Hopf algebra $A = \mathrm{Pol}(SU_q(2))$ [14]

The algebra of polynomials $A = \mathrm{Pol}(SU_q(2))$ is the Hopf $*$-algebra or real quantum group $SU_q(2)$. The coordinate functions π_{ij} are given by

$$\pi_{ij}(d^{\frac{1}{2}}) = \pi_{ij}\begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} = d^{\frac{1}{2}}_{ij} \tag{A.1}$$

where $d^{\frac{1}{2}}$ is the matrix of the fundamental unitary irreducible co-representation of the Hopf algebra A. The co-product Δ, antipode S and co-unit ε act as

$$\Delta d^{\frac{1}{2}}_{ij} = d^{\frac{1}{2}}_{ik} \otimes d^{\frac{1}{2}}_{kj} \tag{A.2}$$

$$S(d^{\frac{1}{2}}) = \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix} \tag{A.3}$$

$$\varepsilon(d^{\frac{1}{2}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.4}$$

Appendix B. Harmonic analysis on the co-set space $M = A/K$ [14]

B.1. The Cartan decomposition of A

The quantum group $K = \mathrm{Pol}(U(1))$ is the Hopf algebra with coordinate functions t and t^{-1}

$$\Delta_U(t^{\pm}) = t^{\pm} \otimes t^{\pm} \qquad S_U(t^{\pm}) = t^{\mp} \qquad \varepsilon_U(t^{\pm}) = 1. \tag{B.1}$$

The Hopf algebra K is the subgroup of the quantum group A defined by the surjective Hopf algebra homomorphism

$$\psi_k\begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \tag{B.2}$$

The left and right unitary co-representation of K in A is given by the homomorphisms

$$L_K = (\psi_k \otimes \mathrm{id}) \circ \Delta \qquad R_K = (\mathrm{id} \otimes \psi_k) \circ \Delta. \tag{B.3}$$

The subspaces $A[j, i]$; $j, i \in Z$ defined by

$$A[j, i] = \{x \in A : L_K(x) = t^j \otimes x \qquad R_K(x) = x \otimes t^i\} \tag{B.4}$$

form the basis of the Hopf algebra A

$$A = \sum_{j,i \in Z} \oplus A[j, i]. \tag{B.5}$$

The quantum co-set space $M = A/K$ and two-sided co-set space $H = K\backslash A/K$ are the subspaces of A defined as

$$M = \sum_{j, \in Z} \oplus A[0, j] \qquad H = A[0, 0]. \tag{B.6}$$

The subspace H is generated by $\xi = bb^*$.

B.2. Harmonic analysis on M

The irreducible co-representation of A is constructed in the space M of homogeneous polynomials of degree l

$$T : \Delta M = M \otimes A. \tag{B.7}$$

The basis in M is composed by the elements

$$e^l_j = a^{l+j} b^{l-j}. \tag{B.8}$$

The matrix elements of the irreducible co-representation of the Hopf $*$-algebra are given in terms of $\xi = bb^* \in H$ and $x \in M$ by

$$d^l_{0j}(x) = \lambda^l_j \, {}_2\phi_1(q^{2(j-l)}, q^{2(j+l+1)}, q^{2(j+1)} \mid q^2, q^2\xi)(-qb^*)^j (a^*)^j \qquad j = 0, 1, \ldots, l \tag{B.9}$$

and

$$d^l_{0j}(x) = \lambda^l_j a^{-j} b^{-j} \, {}_2\phi_1(q^{2(-j-l)}, q^{2(-j+l+1)}, q^{2(-j+1)} \mid q^2, q^2\xi)$$
$$j = -l, -l+1, \ldots, 0. \tag{B.10}$$

Here ${}_2\phi_1$ is the q-hypergeometric function and λ^l_j is defined as

$$\lambda^l_j = q^{|j|(|j|-l)} \begin{bmatrix} l \\ |j| \end{bmatrix}^{\frac{1}{2}}_{q^2} \begin{bmatrix} l-|j| \\ |j| \end{bmatrix}^{\frac{1}{2}}_{q^2}. \tag{B.11}$$

The co-representation (B.7) is unitary with respect to the scalar product

$$\langle x \mid y \rangle = \psi(x^* y) \qquad x, y \in M \tag{B.12}$$

where ψ is the invariant integral on A

$$\psi(z) = \int_0^1 \mathrm{d}\xi_{q^2} \, \mathcal{P}(z) \qquad z \in A \tag{B.13}$$

and $\mathcal{P}$ is the projection operator $\mathcal{P} : A[i, j] \to A[0, 0]$. With respect to the invariant integral (B.13) the matrix elements d^l_{0j} satisfy an orthogonality condition

$$\langle d^l_{0i}(x) \mid d^k_{0j}(x) \rangle = [l + \tfrac{1}{2}]^{-1}_q \delta_{ij} \delta_{lk}. \tag{B.14}$$

The matrix elements given in (B.9) and (B.10) form an orthogonal complete set of functions over M. The Fourier transform of any square integrable function $f(x)$, $x \in M$ is given by

$$f(x) = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} \left[l + \frac{1}{2} \right]_q f^l_j d^l_{0j}(x) \tag{B.15}$$

where the coefficients f^l_j are

$$f^l_j = \langle d^l_{0j} \mid f \rangle. \tag{B.16}$$

Appendix C. The Hopf algebra $U(su_q(2))$ [14]

The Hopf $*$-algebra $U(su_q(2))$ is in non-degenerate duality with A. It is generated by the elements

$$\mathcal{E}_\pm,\ k_\pm = q^{\pm H/4} \tag{C.1}$$

satisfying the commutation relations

$$[\mathcal{E}_+, \mathcal{E}_-] = \frac{k_+^2 - k_-^2}{q - q^{-1}} \qquad k_+k_- = k_-k_+ \qquad k_+\mathcal{E}_+k_- = q\mathcal{E}_+ \tag{C.2}$$

and the involution

$$\mathcal{E}_\pm^* = \mathcal{E}_\mp \qquad H^* = H \tag{C.3}$$

respectively. They are the linear functionals on A

$$\mathcal{E}_+(d^{\frac{1}{2}}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{E}_-(d^{\frac{1}{2}}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad k_\pm(d^{\frac{1}{2}}) = \begin{pmatrix} q^{\pm\frac{1}{2}} & 0 \\ 0 & q^{\mp\frac{1}{2}} \end{pmatrix}. \tag{C.4}$$

The extensions of the functionals (C.1) on the whole algebra A are given by

$$\mathcal{E}_\pm(xy) = \mathcal{E}_\pm(x)k_+(y) + k_-(x)\mathcal{E}_\pm(y) \qquad k_\pm(xy) = k_\pm(x)k_\pm(y) \tag{C.5}$$

where $x, y \in A$. By means of (C.4) and (C.5) we can define the representation of $U(su_q(2))$ from the co-representation (B.7) as

$$\hat{P}d^l_{0,j}(x) = P(d^l_{k,j}(g))d^l_{0,k}(x) \tag{C.6}$$

where $P \in U(su_q(2))$. We then have

$$\hat{\mathcal{E}_\pm}d^l_{0j}(x) = ([l+1\mp j]_q[l\pm j]_q)^{\frac{1}{2}}d^l_{0,j\mp 1}(x) \tag{C.7}$$

$$\hat{k_\pm}d^l_{0j}(x) = q^{\mp j}d^l_{0j}(x). \tag{C.8}$$

The element generating the centre of the Hopf algebra $U(su_q(2))$

$$\mathcal{C} = \mathcal{E}_-\mathcal{E}_+ + \left(\frac{qk_- - q^{-1}k_+}{q - q^{-1}}\right)^2 \tag{C.9}$$

satisfies the e-value equation

$$(\hat{\mathcal{C}} - [l + \tfrac{1}{2}]_q^2)d^l_{0j}(x) = 0. \tag{C.10}$$

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